FIRST-ORDER MODEL CHECKING ON NESTED PUSHDOWN TREES IS COMPLETE FOR DOUBLY EXPONENTIAL ALTERNATING TIME

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Abstract. Recently, we showed that first-order model checking on nested pushdown trees can be done in doubly exponential alternating time with linear many alternations. Using the interpretation method of Compton and Henson we give a matching lower bound.

1. Introduction

Nested pushdown trees were first introduced by Alur et al. [1] as a representation of recursive first-order programs. Nested pushdown trees are unfoldings of pushdown graphs expanded by so-called *jump edges*. These edges connect corresponding push and pop operations. If one considers nested pushdown trees as representations of programs, each push corresponds to a function call and each pop corresponds to a function return. Thus, jump edges allow to reason in first-order logic about pre-/post-conditions on function calls and returns. Note that in the usual representation of first-order programs by pushdown graphs, such a kind of reasoning is even not possible in monadic second-order logic. This advantage comes at a price: while monadic second-order logic is decidable on pushdown graphs [5], it is undecidable on nested pushdown trees [1]. But Alur et al. showed that a variant of modal μ-calculus is still decidable. These results turn nested pushdown trees intro an interesting class of graphs from a model-theoretic point of view. The author only knows one other natural class of graphs with these model checking properties, viz., the class of collapsible pushdown graphs. This common behaviour is explained by the fact that nested pushdown trees are uniformly first-order interpretable in collapsible pushdown graphs [4]. Nevertheless, the two classes differ when one considers first-order model checking. First-order model checking on the second level of the collapsible pushdown graph hierarchy is decidable but has nonelementary complexity [4]. In contrast, for nested pushdown trees we proved an $\mathsf{ATIME}(\mathsf{exp}_2(cn), n)$ upper bound for first-order model checking[3].

The aim of this paper is to provide a matching lower bound for the first-order model checking problem on nested pushdown trees. In fact, we present a fixed nested pushdown tree with $\mathsf{ATIME}(\mathsf{exp}_2(cn), cn)$-hard first-order theory. This implies that the first-order model checking problem on nested pushdown trees is also $\mathsf{ATIME}(\mathsf{exp}_2(cn), cn)$-hard. As a byproduct of our proof we also obtain that the set of FO-sentences valid in all nested pushdown trees and the set of FO-sentences satisfied by some nested pushdown tree are both $\mathsf{ATIME}(\mathsf{exp}_2(cn), cn)$-hard. But we do not know of any upper bound for neither of the two sets.

Our tool is the interpretation method of Compton and Henson [2]: there is a family $(\mathcal{T}_3^{2^n})_{n\in\mathbb{N}}$ of classes of trees such that the following holds. If there is a structure $\mathfrak{A}$ and a sequence of interpretation $(I_n)_{n\in\mathbb{N}}$ using a special type of formulas such that I_n interprets $\mathcal{T}_3^{2^n}$ in a $\mathfrak{A}$, then the theory of $\mathfrak{A}$ is $\mathsf{ATIME}(\mathsf{exp}_2(cn), cn)$-hard. Moreover, the interpretation method is closed under composition in the following sense. If there is a family $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of classes of structures and families of interpretations $(I_n)_{n\in\mathbb{N}}, (J_n)_{n\in\mathbb{N}}$ (using formulas of the mentioned special type) such that I_n interprets $\mathcal{T}_3^{2^n}$ in $\mathcal{C}_n$ and J_n interprets $\mathcal{C}_n$ in $\mathfrak{A}$ then the theory of $\mathfrak{A}$ is $\mathsf{ATIME}(\mathsf{exp}_2(cn), cn)$-hard.

In analogy to Compton and Henson's proof of the $\mathsf{ATIME}(\mathsf{exp}(cn), cn)$-hardness of the first-order theory of the binary successor tree (Example 8.3 in [2]) prove the hardness result for model checking on nested pushdown trees in two steps. First, we provide interpretations of $\mathcal{T}_3^{2^n}$ in finite linear orders of length $\mathsf{exp}_2(13n)$ with one unary predicate. Using another family of interpretations we reduce the monadic second-order theories of such orders to the first-order theory of a fixed nested pushdown tree. This interpretation method yields the $\mathsf{ATIME}(\mathsf{exp}_2(cn), cn)$-hardness of the theory of this nested pushdown tree.

1.1. **Related Work.** It follows from the works of Volger [7] and Compton and Henson [2] that first-order model checking on (unfoldings of) pushdown graphs is $\mathsf{ATIME}(\mathsf{exp}(cn), cn)$-complete. Hence, we show that the introduction of jump edges lead to an exponential blow up in the complexity of model checking. Alur et al. [1] studied the model checking problem on nested pushdown trees for MSO and for a variant of modal μ-calculus. The former is undecidable while the latter is EXPTIME-complete. We [4] have shown that for first-order logic extended with the reachability predicate, the model checking is decidable but has nonelementary complexity (the lower bound already holds for (unfoldings of) pushdown graphs and even for the full infinite binary tree). Since nested pushdown trees are tree-automatic [4] it follows from [6, 4] that the extension of first-order logic by infinity quantifier $\exists^\infty$, modulo counting quantifiers $\exists^{n mod m}$ and Ramsey quantifiers Ram^n is decidable on nested pushdownt trees. The model checking procedure obtained from these results has nonelementary complexity.

1.2. **Outline.** In the next section, we fix our notation, especially concerning interpretations and nested pushdown trees. In Section 3 we recall the central result of Compton and Henson, i.e., we present the classes $\mathcal{T}_3^{2^n}$ of trees and recall how interpretation of these classes in a given structure yields lower bounds for the model checking problem. We then provide interpretations of these classes in finite linear orders of size doubly exponential in n in Section 4. In Section 5 the lower bound for model checking on nested pushdown trees is obtained by interpreting these linear orders of doubly exponential size in a fixed nested pushdown tree. Section 6 contains some concluding remarks.

2. Preliminaries

We set $\mathsf{exp}(n) := 2^n$ and $\mathsf{exp}_2(n) := 2^{\mathsf{exp}(n)}$. In the following $\bar{x}, \bar{y}$, etc. will denote *tuples* of variables $\bar{x} = x_1, x_2, \dots, x_n$, $\bar{y} = y_1, y_2, \dots, y_m$. We ommit the specification of the arity of a tuple $\bar{x}$ whenever the arity is arbitrary or is implicitely defined by the way we use the tuple.

By FO we denote *first-order logic* and by MSO we denote *monadic second-order logic*.

2.1. **Interpretations.** In this paper we will use the interpretation of the MSO theory of a family $(\mathcal{C}_n)_{n\in\mathbb{N}}$ of classes of structures in the MSO theory (or FO theory, respectively) of another family $(\mathcal{D}_n)n \in \mathbb{N}$ in order to transfer lower bounds for the MSO model checking problem on $(\mathcal{C}_n)_{n\in\mathbb{N}}$ to the MSO (FO, respectively) model checking on $(\mathcal{D}_n)_{n\in\mathbb{N}}$. In the following we fix two (relational) signatures $\sigma = \{E_1, E_2, \dots, E_m\}$ and τ. For $\varphi(\bar{x}, \bar{y})$ some σ-formula, $\mathfrak{A}$ some σ-structure and $\bar{p} \in \mathfrak{A}$, we write $\varphi^{\mathfrak{A}}(\bar{x}, \bar{p})$ for the relation defined by φ in $\mathfrak{A}$ with parameter $\bar{p}$. This means that $\varphi^{\mathfrak{A}}(\bar{x}, \bar{p}) := \{\bar{a} \in \mathfrak{A} : \mathfrak{A} \models \varphi(\bar{a}, \bar{p})\}$.

Definition 2.1. Let $(\mathcal{C}_n)_{n\in\mathbb{N}}$ be a family of classes of σ-structures, $(\mathcal{C}'_n)_{n\in\mathbb{N}}$ a family of classes of τ-structures and L either MSO or FO. Furthermore, let x and y be first-order variables and let $\bar{x}_{E_i}$ be a k-tuple of first-order variables where k is the arity of E_i. Let $(I_n)_{n\in\mathbb{N}}$ be a sequence such that

$$I_n = (\delta^n(x, y), (\varphi^n_{E_i}(\bar{x}_{E_i}, y))_{E_i\in\sigma})$$

is an $(m+1)$-tuple of $L[\tau]$-formulas. We call $(I_n)_{n\in\mathbb{N}}$ an *L-to-L-interpretation* of $(\mathcal{C}_n)_{n\in\mathbb{N}}$ in $(\mathcal{C}'_n)_{n\in\mathbb{N}}$ if for each $\mathfrak{A} \in \mathcal{C}_n$, there is a $\mathfrak{B} \in \mathcal{C}'_n$ and some $b \in \mathfrak{B}$ such that

$$\mathfrak{A} \simeq ((\delta^n)^{\mathfrak{B}}(x, b), ((\varphi^n_{E_i})^{\mathfrak{B}}(\bar{x}_{E_i}, b))_{E_i\in\sigma}).$$

Let $(I_n)_{n\in\mathbb{N}}$ be a sequence such that

$$I_n = (\delta^n(x, y), (\varphi^n_{E_i}(\bar{x}_{E_i}, y))_{E_i\in\sigma}, \varphi^n_{\in}(x, z, y))$$

is an $(m+2)$-tuple of $FO[\tau]$-formulas. We call $(I_n)_{n\in\mathbb{N}}$ an MSO-to-FO-interpretation of $(\mathcal{C}_n)_{n\in\mathbb{N}}$ in $(\mathcal{C}'_n)_{n\in\mathbb{N}}$ if for each $\mathfrak{A} \in \mathcal{C}'_n$ there are $\mathfrak{B} \in \mathcal{C}_n$ and $b \in \mathfrak{B}$ such that

$$\mathfrak{A} \simeq ((\delta^n)^{\mathfrak{B}}(x, b), ((\varphi^n_{E_i})^{\mathfrak{B}}(\bar{x}_{E_i}, b))_{E_i\in\sigma})$$

and $\varphi^{n\,\mathfrak{B}}_{\in}(x, b', b)$ ranges over all subsets of $\delta^{n\,\mathfrak{B}}(x, b)$ as b' ranges over the universe of $\mathfrak{B}$.[1]

Remark 2.2. For a fixed structure $\mathfrak{A}$ we write "$(I_n)_{n\in\mathbb{N}}$ is an interpretation of $(\mathcal{C}_n)_{n\in\mathbb{N}}$ in $\mathfrak{A}$" as an abbreviation for "$(I_n)_{n\in\mathbb{N}}$ is an interpretation of $(\mathcal{C}_n)_{n\in\mathbb{N}}$ in $(\mathcal{D}_n)_{n\in\mathbb{N}}$ where $\mathcal{D}_n = \{\mathfrak{A}\}$ for all $n \in \mathbb{N}$".

2.2. **Nested Pushdown Trees.** Nested pushdown trees are the unfoldings of the configuration graphs of pushdown systems with an added *jump relation* that connects every push-with the corresponding pop-operations.

Definition 2.3. A *pushdown system* is a 5-tuple $\mathcal{P} = (Q, \Sigma, \Gamma, \Delta, (q_0, \bot))$ with a finite set of states Q, a finite set of stack symbols Σ, a transition alphabet Γ, an initial configuration $(q_0, \bot) \in Q \times \Sigma$ and a transition relation

$$\Delta \subseteq Q \times \Sigma \times \Gamma \times Q \times (\{\mathsf{pop}, \mathsf{id}_{\Sigma+}\} \cup \{\mathsf{push}_\sigma : \sigma \in \Sigma\}).$$

[1] This condition means that $(I_n)_{n\in\mathbb{N}}$ is an FO-to-FO-interpretation of $(\mathcal{C}_n)_{n\in\mathbb{N}}$ in $(\mathcal{C}'_n)_{n\in\mathbb{N}}$ if we forget about the formulas $\varphi^n_{\in}$ and $\varphi^n_{\in}$ allows to transfer set quantification in $\mathcal{C}_n$ into element quantification in $\mathcal{C}'_n$ by replacing $\exists X$ with $\exists x\delta(x, y)$ and Xz by $\varphi^n(z, x, y)$

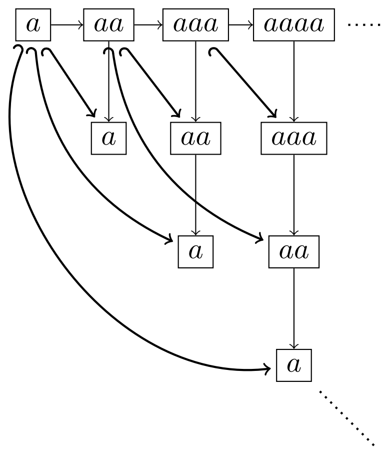

Figure 1: Nested pushdown tree with undecidable MSO theory.

Let $w \in \Sigma^+$ and $\sigma \in \Sigma$. We call w a stack. A *configuration* of $\mathcal{P}$ is a pair $(q, w) \in Q \times \Sigma^+$ of a state and a nonempty stack. We define the stack operations for all $\sigma \in \Sigma$ and $w \in \Sigma^+$ by

$$\mathsf{push}_\sigma(w) := w\sigma \text{ and } \mathsf{pop}(w\sigma) = w.$$

We define labelled transitions $\xrightarrow{\gamma}$ on the set of configurations of $\mathcal{P}$ for $\gamma \in \Gamma$ as follows: $(q, w\sigma) \xrightarrow{\gamma} (p, v)$ if there is some $(q, \sigma, \gamma, p, \mathrm{op}) \in \Delta$ such that $v = \mathrm{op}(w\sigma)$. We use $\rightarrow$ as abreviation of $\bigcup_{\gamma \in \Gamma} \xrightarrow{\gamma}$.

A *run* r of $\mathcal{P}$ is a sequence $c_0 \xrightarrow{\gamma_1} c_1 \xrightarrow{\gamma_2} c_2 \xrightarrow{\gamma_3} \dots \xrightarrow{\gamma_n} c_n$. We denote by $\preceq$ the prefix order on the set of runs. We call r a run from c_0 to c_n and say that the *length* of r is $\mathsf{length}(r) := n$.

Definition 2.4. Let $\mathcal{P} = (Q, \Sigma, \Delta, (q_0, \bot))$ be a pushdown system. Then the *nested pushdown tree* generated by $\mathcal{P}$ is the structure $\mathsf{NPT}(\mathcal{P}) := (R, (\xrightarrow{\gamma})_{\gamma \in \Gamma}, \hookrightarrow)$ defined as follows:

- R is the set of runs starting in $(q_0, \bot)$,
- $\xrightarrow{\gamma}$ connects a run r_1 with a run r_2 if r_2 extends r_1 by exactly one $\xrightarrow{\gamma}$ transition at the end, and
- the so-called *jump-relation* $\hookrightarrow$ connects $r_1 \in R$ with $r_2 \in R$ if $r_1 \preceq r_2$, the final stack of r_1 and r_2 is the same word w and all stacks between have w as proper prefix.

Remark 2.5. Let $r_1 \preceq r_2$ be runs. Then $r_1 \hookrightarrow r_2$ holds if and only if r_2 extends r_1 by some run r that starts with some push_σ transition and ends with a pop transition that removes this σ from the stack again.

Example 2.6. Figure 1 shows the nested pushdown tree induced by the transition relation $(q_0, a, \gamma_1, \mathsf{push}_a, q_0), (q_0, a, \gamma_2, \mathsf{pop}, q_0)\}$. The MSO theory of this nested pushdown tree is undecidable because the symmetric transitive closure of γ_2-edges is the "same column relation" and the symmetric transitive closure of the jump edges is the "same diagonal relation" whence the half grid $\{(n, m) : m > n\}$ with right and downward edges is MSO interpretable in this graph. Standard arguments reduce undecidable tiling problems to the MSO theory of this half grid.

The previous example shows that the MSO theories of nested pushdown trees are undecidable in general. But the FO theories are uniformly decidable. In [3] (Theorem 2), we provided an FO model checking algorithm on nested pushdown trees. Even though we only claimed it was an 2-EXPSPACE algorithm, the proof reveals the following fact.

Theorem 2.7. FO *model checking on nested pushdown trees is in* $\mathsf{ATIME}(\mathsf{exp}_2(cn), n)$.

3. The Interpretation Method

In this section, we recall some results of Compton and Henson [2] that we are going to use in the following. We first present the classes $\mathcal{T}_3^{2^n}$ of trees. Then we recall the necesary results relating interpretations of these classes to $\mathsf{ATIME}(\mathsf{exp}_2(cn), cn)$-hardness of the model checking problem.

3.1. Classes of Special Trees.

A tree is a finite, prefix closed subset of $\mathbb{N}^*$ together with the successor relation $S := \{(x,y) \in \mathbb{N}^* : \exists z \in \mathbb{N} \quad y = xz\}$. For $\mathfrak{T}$ some tree we call $|\mathfrak{T}|$ its *size*, i.e. $|\mathfrak{T}|$ is the number of elements in $\mathfrak{T}$. For $\mathfrak{T} = (T, S)$ some tree and $d \in T$, we denote by $\mathfrak{T}_d$ the subtree rooted at d, i.e., $\mathfrak{T}_d = (T_d, S)$ where $T_d = \{x \in \mathbb{N}^* : dx \in T\}$. Note that for each tree $\mathfrak{T} = (T, S)$ the set $\mathbb{N} \cap T$ is the set of children of the root.

Definition 3.1. Let $\mathcal{T}_0^{2^n}$ be the class of the tree of depth 0, i.e., the class containing $(\{\varepsilon\}, S)$. Inductively, let $\mathcal{T}_{n+1}^{2^n}$ be the class of trees $\mathfrak{T} = (T, S)$ such that $\mathfrak{T}_d \in \mathcal{T}_n^{2^n}$ for all $d \in \mathbb{N} \cap T$ and for each $d \in \mathbb{N} \cap T$ there are at most 2^n many pairwise distinct elements $d' \in \mathbb{N} \cap T$ such that $\mathfrak{T}_{d'} \simeq \mathfrak{T}_d$, i.e., there are at most 2^n isomorphic maximal proper subtrees of $\mathfrak{T}$.

We will exclusively deal with $\mathcal{T}_3^{2^n}$ in this paper. The following lemma summarises some combinatorial properties of this class.

Lemma 3.2. $\mathcal{T}_3^{2^n}$ *contains at most* $(2^n+1)^{(2^n+1)^{2^n+1}}$ *many trees up to isomorphism. A tree* $\mathfrak{T} \in \mathcal{T}_3^{2^n}$ *has size at most* $2^{2^{12n}}$.

Proof. The first claim is by induction: Note that $\mathcal{T}_0^{2^n}$ consists of only one tree. Each element of $\mathcal{T}_{i+1}^{2^n}$ is determined by the number of maximal proper subtrees of each isomorphism type. Since there are at most 2^n maximal proper subtrees of the same isomorphism type, there are at most $(2^n+1)^m$ many nonisomorphic trees in $\mathcal{T}_{i+1}^{2^n}$ where m is the number of distinct trees in $\mathcal{T}_i^{2^n}$ up to isomorphism. The first claim follows by induction.

Let $f(i)$ be the number of nonisomorphic trees in $\mathcal{T}_i^{2^n}$ and $g(i)$ the maximal number of nodes of a tree in $\mathcal{T}_i^{2^n}$. Note that the size of a tree $\mathfrak{T} \in \mathcal{T}_{i+1}^{2^n}$ is bounded by $1 + 2^n \cdot f(i) \cdot g(i)$. We conclude that

- for $\mathfrak{T} \in \mathcal{T}_1^{2^n}$, $|\mathfrak{T}| \leq 1 + 2^n \cdot 1 \cdot 1 = 2^n + 1$,
- for $\mathfrak{T} \in \mathcal{T}_2^{2^n}$, $|\mathfrak{T}| \leq 1 + 2^n \cdot (2^n+1) \cdot (2^n+1) \leq 2^{6n}$, and
- for $\mathfrak{T} \in \mathcal{T}_3^{2^n}$, $|\mathfrak{T}| \leq 1 + 2^n \cdot (2^n+1)^{2^n+1} \cdot 2^{6n} \leq 1 + 2^{7n+(n+1)\cdot(2^n+1)} \leq 2^{2^{12n}}$

□

3.2. **Iterative Definitions and Prescribed Sets.** The relevance of $(\mathcal{T}_3^{2^n})_{n\in\mathbb{N}}$ for this paper stems from the fact that certain interpretations of these classes in a fixed structure $\mathfrak{A}$ allow to establish $\mathsf{ATIME}(\mathsf{exp}_2(cn), cn)$-hardness of the theory of $\mathfrak{A}$. Before we can specify the kind of interpretations that one may use for this purpose, we have to recall Comton and Henson's notion of *explicit* and *iterative* definitions. Let MSO^* denote the extension of MSO by exlicit and iterative definitions. MSO^* is defined using the formation rules of MSO and the rules that

- for $\psi, \varphi \in \mathsf{MSO}^*$ and P some free variable of ψ, the formula $[P = \varphi]\psi$ is in MSO^* where $[P = \varphi]$ is called an explicit definition, and
- for $\psi, \varphi \in \mathsf{MSO}^*$ and P some free variable of ψ and φ, the formula $[P = \varphi]_n\psi$ is in MSO^* where $[P = \varphi]_n$ is called an iterative definition.

The semantics of MSO^* is defined using the semantics of MSO and the following rules.[2]

- Let $\mathfrak{A}$ be some structure with domain A. Let P_φ be the predicate that contain a tuple $\bar{a} \in A$ iff $\mathfrak{A} \models \varphi(\bar{a})$. Now $\mathfrak{A} \models [P = \varphi]\psi$ iff $\mathfrak{A}, P_\varphi \models \psi$, i.e. if $\mathfrak{A}$ is a model of ψ where each occurrence of P is replaced by the relation defined by φ.
- Let $\mathfrak{A}$ be some structure. $\mathfrak{A}$ satisfies an iterative definition $[P = \varphi]_0\psi$ if $\mathfrak{A}$ satisfies ψ where each occurrence of P is replaced by $\exists x x \neq x$, i.e., by a sentence which is false in every structure. $\mathfrak{A}$ satisfies an iterative definition $[P = \varphi]_{n+1}\psi$ iff $\mathfrak{A}$ satisfies $[P = [P = \varphi]_n\varphi]\psi$.

Adding explicit and iterative definitions do not increase the expressive power of MSO or FO but allow for more succint definitions.

Definition 3.3. We call a set M of MSO^*-formulas a *prescribed set*[3] if there is some $l \in \mathbb{N}$ such that each $\varphi \in M$ is of the form

$$[P_1 = \varphi_1]_{n_1}[P_2 = \varphi_2]_{n_2} \dots [P_k = \varphi_k]_{n_k}\psi$$

for some $k \in \mathbb{N}$ and $n_1, n_2, \dots, n_k \in \mathbb{N}$ where each φ_j is an MSO formula of size at most l whose only free set variable is P_j and ψ is an MSO formula in prenex normal form, i.e., ψ is quantifier free except for a block of quantifiers at the beginning. Fix some prescribed set M and an interpretation $I = (I_n)_{n\in\mathbb{N}}$. We say I *is formed from* M and – abusing notation – write $I \subseteq M$ if for each $n \in \mathbb{N}$ and for each formula φ in I_n there is an equivalent formula $\varphi' \in M$ of size linear in n (where the subscript n_i occuring in iterative definitions $[P_i = \varphi_i]_{n_i}$ are written in unary notation, i.e., a subscript n_i counts as a string of length n_i.

In the rest of this paper, the interpretations we define are always formed from some prescribed set. Since iterative definitions do not increase the expressive power of FO or MSO and since we aim at showing that the formulas used have equivalent versions in some prescribed set, we will use iterative definitions when defining MSO-to-MSO- or MSO-to-FO-interpretations.

3.3. **Lower Bounds for Model Checking via Interpretations.** One of the central results of Compton and Henson (Corollary 7.5 in [2]) is that interpretations $(I_n)_{n\in\mathbb{N}}$ formed from some prescribed set M can be used to obtain lower bounds on the model checking

[2]In the following rules we supress the occurrence of free variables; these are handled as usual.

[3]Compton and Henson's definition of prescribed sets is more general, but the special cases defined here suffice for our purpose.

problem. We only state a simplified version of the more general result which is sufficient for our purposes.

Lemma 3.4 ([2])**.** *If there is some prescribed set M and $(I_n)_{n\in\mathbb{N}} \subseteq M$ is an* MSO*-to-*FO*-interpretation of $(\mathcal{T}_3^{2^n})_{n\in\mathbb{N}}$ in a structure $\mathfrak{A}$, then the* FO *theory of $\mathfrak{A}$ is* $\mathsf{ATIME}(\mathsf{exp}_2(cn), cn)$*-hard.*

Remark 3.5. In fact, Compton and Henson prove a stronger result. Under the assumptions of the lemma, $\mathfrak{A}$ has a *hereditary lower bound* of $\mathsf{ATIME}(\mathsf{exp}_2(cn), cn)$. This means that for T the FO-theory of $\mathfrak{A}$, and every subset $\Phi \subseteq T$ the sets $\mathsf{SAT}(\Phi)$ and $\mathsf{VAL}(\Phi)$ are $\mathsf{ATIME}(\mathsf{exp}_2(cn), cn)$-hard where $\mathsf{SAT}(\Phi)$ denotes the set of FO sentences that are satisfiable in some model of Φ and $\mathsf{VAL}(\Phi)$ denotes the set of FO sentences that are valid in every model of Φ.

Corollary 3.6. *Under the same assumptions as in the lemma, the* FO *model checking problem on any class $\mathcal{C}$ such that $\mathfrak{A} \in \mathcal{C}$ is* $\mathsf{ATIME}(\mathsf{exp}_2(cn), cn)$*-hard.*

The use of the previous lemma is further facilitated since interpretations formed from prescribed sets are closed under composition.

Lemma 3.7 ([2])**.** *If M, M' are prescribed sets and $(I_n)_{n\in\mathbb{N}} \subseteq M, (J_n)_{n\in\mathbb{N}} \subseteq M'$ are families of interpretations such that $(I_n)_{n\in\mathbb{N}}$ is an* MSO*-to-*MSO*-interpretation of $(\mathcal{C}_n)_{n\in\mathbb{N}}$ in $(\mathcal{D}_n)_{n\in\mathbb{N}}$ and $(J_n)_{n\in\mathbb{N}}$ is an* MSO*-to-*FO*-interpretation of $(\mathcal{D}_n)_{n\in\mathbb{N}}$ in $(\mathcal{E}_n)_{n\in\mathbb{N}}$, then there is a prescribed set M'' and an* MSO*-to-*FO*-interpretation $(K_n)_{n\in\mathbb{N}} \subseteq M''$ of $((C)_n)_{n\in\mathbb{N}}$ in $(\mathcal{E}_n)_{n\in\mathbb{N}}$.*

4. Interpretation of Trees in Linear Orders

Following the ideas of Compton and Henson (Examples 8.1, 8.6, and 8.8 in [2]) we first provide interpretations of the classes $(\mathcal{T}_3^{2^n})_{n\in\mathbb{N}}$ in the classes $(\mathcal{L}_{13n})_{n\in\mathbb{N}}$ of linear orders of size exactly $2^{2^{13n}}$ with unary predicate P. We identify such a linear order $\mathfrak{L}$ with a bitstring of the same length, where we interpret P as indicator of 1's in the bitstring. Fix a tree $\mathfrak{T}$. We use a string of the form 0^i1 in order to represent a node of depth i and encode $\mathfrak{T}$ by traversing $\mathfrak{T}$ in-order and code every node with the corresponding representation. The following function f does this encoding. For $k \in \mathbb{N}$ and $\mathfrak{T}$ some tree of depth at most k, set

$$f(\mathfrak{T}, k) := 0^{k-1}1f(\mathfrak{T}_{i_1}, k-1)f(\mathfrak{T}_{i_2}, k-1)f(\mathfrak{T}_{i_3}, k-1)\dots f(\mathfrak{T}_{i_k}, k-1)$$

where $i_j \in \mathbb{N}$ is the j-th element of $\mathbb{N}$ such that $i_j \in \mathfrak{T}$.

For $\mathfrak{T} \in \mathcal{T}_3^{2^n}$, its depth is bounded by 3 and $\mathfrak{T}$ has at most $2^{2^{12n}}$ nodes. Thus, $f(\mathfrak{T}, 3)$ has length at most $4 \cdot 2^{2^{12n}} \leq 2^{2^{13n}}$ By padding 0's in the end, we obtain bitstrings of length $2^{2^{13n}}$ that encodes $\mathfrak{T}$. As mentioned before, we identify this bitstring with a linear order in $\mathcal{L}_{13n}$.

In Example 8.1 of [2], Compton and Henson show that the classes $\mathcal{T}_3^{2^n}$ can be recovered from these encodings with MSO-to-MSO-interpretations.

Lemma 4.1. *There is a prescribed set M such that there is an* MSO*-to-*MSO*-interpretations $(I_n)_{n\in\mathbb{N}} \subseteq M$ of $(\mathcal{T}_3^{2^n})_{n\in\mathbb{N}}$ in $(\mathcal{L}_{13n})_{n\in\mathbb{N}}$.*

Proof. Let

$$Pr(x,y,z) := \quad x < y \wedge (x < z \rightarrow y \leq z) \text{ and}$$
$$\psi(x,y,Q) := \quad \exists x_1 \exists y_1 \forall z ((z \geq x \vee (P(x_1) \wedge Pr(x_1,x,z))) \wedge ((z \geq y \vee (P(y_1) \wedge Pr(y_1,y,z))))$$
$$\vee (Pr(x_1,x,z) \wedge Pr(y_1,y,z) \wedge \neg P(x_1) \wedge \neg P(y_1) \wedge Q(x_1,y_1)))$$

In ψ, $Pr(x,y,z)$ is used to express that x is the direct predecessor of y. $[Q=\psi]_{n+1}$ defines those tuples (a,b) where the number of consecutive 0's preceeding a is equal to the number of consecutive 0's preceeding b and this number is at most n. Then

$$\varphi_E := [Q=\psi]_4 \exists x_1 \forall z' \forall z$$
$$Pr(x_1,x,z') \wedge x < y \wedge P(x) \wedge P(y) \wedge \neg P(x_1) \wedge Q(x_1,y) \wedge (x < z < y \rightarrow \neg Q(x,z))$$

says that y and the predecessor of x have the same number of consecutive preceeding 0's (up to 3) and no element in between x and y has the same number of consecutive preceeding 0's as x (up to 3). On a linear order that stems from the encoding $f(\mathfrak{T})$ for some $\mathfrak{T} \in \mathcal{T}_3^{2^n}$, φ_E interprets exactly the edge relation of $\mathfrak{T}$. Setting $I_n := (\delta(x), \varphi_E(x,y))$, $(I_n)_{n\in\mathbb{N}}$ is an MSO-to-MSO-interpretation of $(\mathcal{T}_3^{2^n})_{n\in\mathbb{N}}$ in $(\mathcal{L}_{13n})_{n\in\mathbb{N}}$. □

5. Reduction of Linear Orders to NPT

Due to Lemmas 3.4, 3.7 and 4.1, it suffices to provide an MSO-to-FO-interpretation of $\mathcal{L}_{13n}$ in some fixed nested pushdown tree in order to show the $\mathsf{ATIME}(\mathsf{exp}_2(cn), cn)$-hardness of FO model checking on NPT. In the rest of this paper, we consider the fixed pushdown system

$$\mathcal{S} := (Q, \Sigma, \Gamma, \Delta, (q_0, a))] \text{ with}$$
$$Q = \{q_0, q_1\}, \Sigma = \{a\},$$
$$\Gamma = \{+_0, +_1, -_0, -_1\} \text{ and}$$
$$\Delta = \{(q_i, a, +_j, \mathsf{push}_a, q_j), (q_i, a, -_j, \mathsf{pop}, q_j) : i,j \in \{0,1\}\}.$$

In each configuration $\mathcal{S}$ nondeterministically changes to state q_0 or q_1 and performs a push or a pop operation. This means that the runs of $\mathcal{S}$ are all possible runs of pushdown systems with 2 states 1 stack symbol. The nested pushdown system generated by $\mathcal{S}$ is depicted in Figure 2.

In the following, we show that $\mathsf{NPT}(\mathcal{S})$ contains nodes a that have $2^{2^{13n}}$ pairwise distinct ancestors each of which is connected to a by a path of length 2^{13n} whose edges consist of $\xrightarrow{-i}$ and $\hookrightarrow$, i.e., $|\{b \in \mathsf{NPT}(\mathcal{S}) : \mathsf{NPT}(\mathcal{S}) \models \varphi_p^{=2^{13n}}(b,a)\}| = \mathsf{exp}_2(13n)$. Such a are used to represent elements from $\mathcal{L}_{13n}$ as follows: each of the $2^{2^{13n}}$ ancestors of a represent one element of $\mathcal{L}_{13n}$. The order is given by the ancestor relation which can be expressed in linear size using the formula $\varphi_p^{\leq 2^{13n}}$. Furthermore, the unary predicate P contains those nodes that are in state q_1. This yields an FO-to-FO-interpretation of $\mathcal{L}_{13n}$ in $\mathsf{NPT}(\mathcal{S})$. We extend this interpretation to an MSO-to-FO-interpretation as follows. Given nodes a_1 and a_2 representing linear orders with one unary predicate P_{a_1} and P_{a_2} respectively, the formula $\varphi_e^{=2^{13n}}(b_1,a_1,b_2,a_2)$ is satisfied if and only if there is some $j \leq 2^{2^{13n}}$ such that b_1 is the j-th element of the order induced by a_1 and b_2 is the j-th element of the order induced by a_2. Using this fact, we can express "a is the j-th node of the order induced by x and the j-th element of the order induced by y is in P_y". Thus, given a fixed $g \in \mathsf{NPT}(\mathcal{S})$ representing a

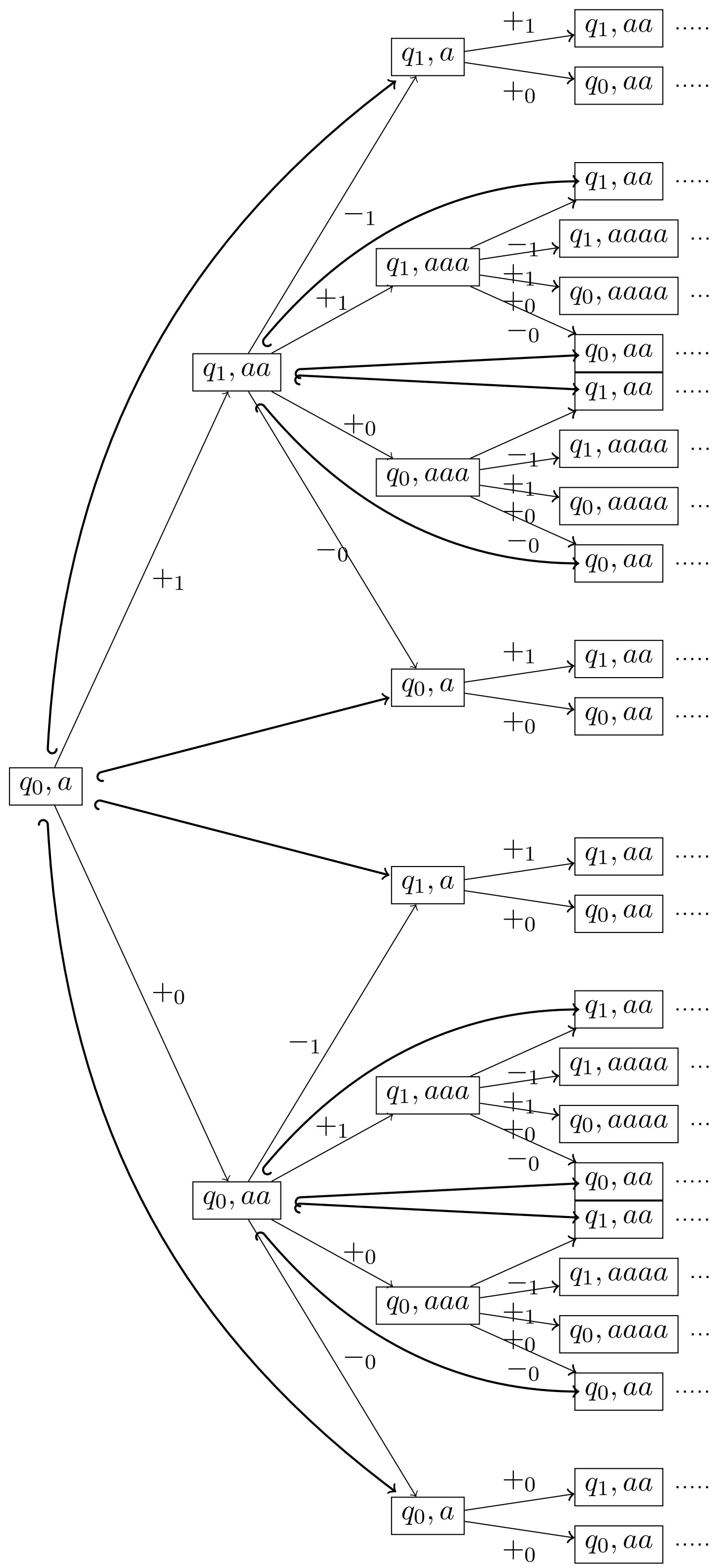

Figure 2: Nested pushdown tree $\mathsf{NPT}(\mathcal{S})$.

linear order in $\mathcal{L}_{13n}$, quantification over representations of linear orders in $\mathcal{L}_{13n}$ is the same as quantification over monadic predicates in the order induced by g.

5.1. **Short Formulas for Paths of Exponential Lenth.** Aiming at the interpretation of $\mathcal{L}_{13n}$ in $\mathsf{NPT}(\mathcal{S})$, we define formulas $\varphi_{\overline{p}}^{\leq 2^{13n}}$, $\varphi_{\overline{p}}^{=2^{13n}}$, $\varphi_{\overline{e}}^{\leq 2^{13n}}$, and $\varphi_{\overline{e}}^{=2^{13n}}$ talking about paths in $\mathsf{NPT}(\mathcal{S})$. $\varphi_{\overline{p}}^{\leq 2^{13n}}(x,y)$ expresses that there is a path of length at most 2^{13n} from x to y using only pop transitions and jump edges. $\varphi_{\overline{p}}^{=2^{13n}}(x,y)$ is the variant for a path of length exactly 2^{13n}. $\varphi_{\overline{p}}^{\leq 2^{13n}}(x_1,x_2,y_1,y_2)$ expresses that there is a path of length $k \leq 2^{13n}$ from x_1 to x_2 and a path of length k from y_1 to y_2 such that both paths only consist of jump edges and pop transitions and the i-th edge in one of the paths is a jump edge if and only if the i-th edge in the other is also a jump edge. Analogously, the i-th edge in one path is a pop if and only if the i-th edge in the other path is a pop.

Definition 5.1. Define by induction on m the following formulas:

$$\begin{aligned}
\psi_1(x,y,Q) :=\ & \exists z\,(Qxz \wedge Qzy) \vee x=y \vee x \xrightarrow{-0} y \vee x \xrightarrow{-1} y \vee x \hookrightarrow y\\
\varphi_{\overline{p}}^{\leq 2^m}(x,y) :=\ & [Q=\psi_1]_{m+1}Qxy\\
\psi_2(x,y,R) :=\ & \exists z \forall z' \forall z''\,(Rxz \wedge Rzy) \vee \Big(\neg Rz'z'' \wedge (x \xrightarrow{-0} y \vee x \xrightarrow{-1} y \vee x \hookrightarrow y)\Big)\\
\varphi_{\overline{p}}^{=2^m}(x,y) :=\ & [R=\psi_2]_{m+1}Rxy\\
\psi_3(x_1,x_2,y_1,y_2,S) :=\ & \exists z \exists z'\,(Sx_1zy_1z' \wedge Szx_2z'y_2)\\
& \vee (x_1 = x_2 \wedge y_1 = y_2) \vee (x_1 \hookrightarrow x_2 \wedge y_1 \hookrightarrow y_2) \vee\\
& \Big((x_1 \xrightarrow{-0} x_2 \vee x_1 \xrightarrow{-1} x_2) \wedge (y_1 \xrightarrow{-0} y_2 \vee y_1 \xrightarrow{-1} y_2)\Big)\\
\varphi_{\overline{e}}^{\leq 2^m}(x_1,x_2,y_1,y_2) :=\ & [S=\psi_3]_{m+1}Sx_1x_2y_1y_2\\
\psi_4(x_1,x_2,y_1,y_2,T) :=\ & \exists z \exists z' \forall t \forall u \forall v \forall w\,(Tx_1zy_1z' \wedge Tzx_2z'y_2)\\
& \vee \Big(\neg Ttuvw \wedge \big((x_1 \hookrightarrow x_2 \wedge y_1 \hookrightarrow y_2) \vee\\
& ((x_1 \xrightarrow{-0} x_2 \vee x_1 \xrightarrow{-1} x_2) \wedge (y_1 \xrightarrow{-0} y_2 \vee y_1 \xrightarrow{-1} y_2))\big)\Big)\\
\varphi_{\overline{e}}^{=2^m}(x_1,x_2,y_1,y_2) :=\ & [T=\psi_4]_{m+1}Tx_1x_2y_1y_2
\end{aligned}$$

5.2. **Nodes with many Ancestors.** In this section, we prove that there are formulas of size linear in n that define subsets of $\mathsf{NPT}(\mathcal{S})$ of size $\mathsf{exp}_2(13n)$ that can be interpreted as bitstrings of the same size. We start with an auxiliary lemma that allows to separate different ancestors of a given node of $\mathsf{NPT}(\mathcal{S})$. The lemma says the following: Given two paths p_1, p_2 in $\mathsf{NPT}(\mathcal{S})$ that consist of jump edges and pop transitions such that p_1 and p_2 end in the same node $c \in \mathsf{NPT}(\mathcal{S})$ such that p_1 ends in a jump edge and p_2 ends in a pop transition, then p_1 starts in an ancestor of the first node of p_2.

Lemma 5.2. *Let, $a, a'b, b', c$ be nodes of $\mathsf{NPT}(\mathcal{S})$ and $m \in \mathbb{N}$. If*

$$\mathsf{NPT}(\mathcal{S}) \models \varphi_{\overline{p}}^{\leq 2^m}(a,b) \wedge \varphi_{\overline{p}}^{\leq 2^m}(a',b') \wedge b' \to c \wedge b \hookrightarrow c$$

then $a \preceq b \prec a' \preceq b' \prec c$.

Proof. The nontrivial claim is that $b \prec a'$. If $x \xrightarrow{-j} y$ for some $j \in \{0,1\}$, then x has a bigger stack than y. Furthermore, if $x \hookrightarrow y$ then the stacks of x and y agree and all $x \prec z \prec y$ have bigger stacks.

Applying this observation to $b \hookrightarrow c$, we obtain that the predecessor b' of c is connected via a pop transition to c. Thus, the stacks of b and c agree while the stack of b' is bigger than that of c. Furthermore, between a' and b' all stacks are at least as big as the stack of b'. Since $b \prec c$ and its stack is smaller than that of b', one concludes that $b \prec a'$. □

Using the previous lemma inductively, one concludes that going backward different pop and jump-edge paths lead to different nodes. One formalisation of this claim is the following corollary.

Corollary 5.3. *Let $a, b, c \in \mathsf{NPT}(\mathcal{S})$ be nodes such that $\mathsf{NPT}(\mathcal{S}) \models \varphi_p^{\leq 2^{13n}}(a,c) \wedge \varphi_p^{\leq 2^{13n}}(b,c)$. Either $\mathsf{NPT}(\mathcal{S}) \models \varphi_e^{\leq 2^{13n}}(a,c,b,c)$ or $a \neq b$.*

Definition 5.4. Set $\varphi_{lin}^n(x) := \forall y \exists z (\varphi_p^{\leq 2^{13n}}(y,x) \rightarrow (z \hookrightarrow y))$. For each $a \in \mathsf{NPT}(\mathcal{S})$, we set $D_a^n := \left\{ b \in \mathsf{NPT}(\mathcal{S}) : \mathsf{NPT}(\mathcal{S}) \models \varphi_p^{=2^{13n}}(b,a) \right\}$.

Since $z \hookrightarrow y$ implies that y has a direct predecessor $z' \rightarrow y$ such that the transition from z' to y is a pop-transition, $\varphi_{lin}^n(x)$ is satisfied if and only if for each sequence s of pop-transitions and jump-edges of length $2^{13n}+1$ there is some node y connected to x via a path of form s. Due to Corollary 5.3, all these different sequences lead to different ancestors of x. Thus, if $\varphi_{lin}^n(x)$ holds, the paths to all elements of D_x^n form a full binary tree of depth 2^{13n}. Since D_x^n is the set of leaves of this tree, it contains exactly $\mathsf{exp}_2(13n)$ elements.

Corollary 5.5. *If $\mathsf{NPT}(\mathcal{S}) \models \varphi_{lin}^n(a)$ then $|D_a^n| = \mathsf{exp}_2(13n)$.*

We will now construct elements $a \in \mathsf{NPT}(\mathcal{S})$ that satisfy φ_{lin}^n.

Lemma 5.6. *For each m there is a node $a \in \mathsf{NPT}(\mathcal{S})$ with 2^m many ancestors of distance m. Each of these ancestors is connected to a via some path of length m that only uses jump edges and pop transitions.*

Proof. The proof is by induction on m. In fact, we prove the following stronger claim: for $m \in \mathbb{N}$ and an arbitrary $a_0 \in \mathsf{NPT}(\mathcal{S})$, we can construct a node $a \in \mathsf{NPT}(\mathcal{S})$ with 2^m many ancestors of distance m such that each of these ancestors is a descendant of a_0 and connected to a via some path of length m that only uses $\hookrightarrow$- and pop-edges. Furthermore, a_0 is connected to a via a path of m $\hookrightarrow$-edges.

For $m = 0$ the claim holds trivially by setting $a := a_0$.

Now assume that for some $m \in \mathbb{N}$ the claim holds. Let $a_0 \in \mathsf{NPT}(\mathcal{S})$. Let a_1 be a node in $\mathsf{NPT}(\mathcal{S})$ satisfying the claim with respect to m and a_0. Let a_2 be the unique node such that $a_1 \xrightarrow{+1} a_2$. Let a_3 be a node in $\mathsf{NPT}(\mathcal{S})$ satisfying the claim with respect to m and a_2. Let a be the unique node such that $a_3 \xrightarrow{-1} a$.

Note that a_1 and a_3 are the ancestors of a of distance 1. Each of these has 2^m ancestors of distance m. By Lemma 5.2 these are disjoint whence a has $2 \cdot 2^m = 2^{m+1}$ ancestors at distance $m+1$. Moreover a_0 is connected to a_1 via a $\hookrightarrow$ path of length m and $a_1 \hookrightarrow a$. Thus, a satisfies the claim. □

Remark 5.7. Note that the construction in the previous proof does not rely on the use of the transition $\xrightarrow{+1}$ and $\xrightarrow{-1}$. In each construction step, we can arbitrarily replace $\xrightarrow{+1}$ by $\xrightarrow{+0}$ and $\xrightarrow{-1}$ by $\xrightarrow{-0}$. Hence, the state of each node occurring in the construction can be chosen independently.

Corollary 5.8. *For a fixed string $b \in \{0,1\}^{\mathsf{exp}_2(13n)}$, there is some $a \in \mathsf{NPT}(\mathcal{S})$ such that $\mathsf{NPT}(\mathcal{S}) \models \varphi^n_{lin}(a)$ and the i-th element of D^n_a (w.r.t. $\prec$) is in state q_1 if and only if the i-th bit of b is 1.*

5.3. **Interpretation of Order and Monadic Quantification.** We fix an element a satisfying $\varphi^n_{lin}(a)$. We can interpret every $\hookrightarrow$-edge as 0 and every $\rightarrow$-edge as 1. Using this convention each path p of length 2^{13n} from some ancestor b to a can be interpreted as the 2^n-bit number $\hat{p}$ induced by its transitions. By induction on Lemma 5.2, we obtain that b is the $\hat{p}_b$-th element of D^n_a with respect to $\prec$ for all $b \in D^n_a$ and for p_b the unique path from b to a of length 2^{13n}.

We next present a formula of size linear in n that defines $\prec$ on D^n_a. Afterwards we will show that monadic quantification in linear orders in $\mathcal{L}_{13n}$ can be reduced to first-order quantification in $\mathsf{NPT}(\mathcal{S})$.

Recall that $b \prec b'$ holds for $b, b' \in D^n_a$ if and only if for p the minimal path from b to a and p' the minimal path from b' to a (both of length 2^{13n}) have a common suffix and at the maximal position where p and p' differ, p consists of a jump edge. Note that this implies that p' contains at this position a $\xrightarrow{-j}$-edge for some $j \in \{0,1\}$. Let $I_n = (\delta^n, \varphi^n_<, \varphi^n_P)$ be given by

$$\begin{aligned}
\delta^n(x,y) :=&\quad \varphi^n_{lin}(y) \wedge \varphi_p^{=2^{13n}}(x,y)\\
\varphi^n_<(x_1,x_2,y) :=&\exists z \exists z_1 \exists z_2 (z_1 \hookrightarrow z \wedge z_2 \rightarrow z \wedge \varphi_p^{\leq 2^{13n}}(z,y) \wedge \varphi_p^{\leq 2^{13n}}(x_1,z_1) \wedge \varphi_p^{\leq 2^{13n}}(x_2,z_2)) \wedge\\
&\quad \delta^n(x_1,y) \wedge \delta^n(x_2,y)\\
\varphi^n_P(x,y) :=&\quad \exists z (z \xrightarrow{-1} x \vee z \xrightarrow{+1} x) \wedge \delta^n(x,y)
\end{aligned}$$

Note that for every $a \in \mathsf{NPT}(\mathcal{S})$ with $\mathsf{NPT}(\mathcal{S}) \models \varphi^n_{lin}(y)$, $\delta^{n\,\mathsf{NPT}(\mathcal{S})}(x,a)$ is a set of size $\mathsf{exp}_2(13n)$ that is linearly ordered by $\varphi_<^{n\,\mathsf{NPT}(\mathcal{S})}(x_1,x_2,a)$. Furthermore, $\varphi_P^{n\,\mathsf{NPT}(\mathcal{S})}(x,a)$ selects the subset of nodes of $\delta^{n\,\mathsf{NPT}(\mathcal{S})}(x,a)$ which represent runs that end in state q_1. Due to remark 5.7, for each $\mathfrak{L} \in \mathcal{L}_{13n}$ there is some $a \in \mathsf{NPT}(\mathcal{S})$ such that I_n interprets $\mathfrak{L}$ in $\mathsf{NPT}(\mathcal{S})$. Thus, $(I_n)_{n\in\mathbb{N}}$ is an FO-to-FO-interpretation of $(\mathcal{L}_{13n})_{n\in\mathbb{N}}$ in $\mathsf{NPT}(\mathcal{S})$.

We extend this interpretation to an MSO-to-FO-interpretation. Given some $\mathfrak{L} \in \mathcal{L}_{13n}$ we can identify its domain with the set $\{1,2,\dots,\mathsf{exp}_2(13n)\}$ such that its order coincides with the order of the natural numbers on this set. Given some $a \in \mathsf{NPT}(\mathcal{S})$ such that $\mathsf{NPT}(\mathcal{S}) \models \varphi^n_{lin}(a)$, we identify the linear order $\mathfrak{L}_a$ obtained by I'_n with parameter a with the set $\{n \in \mathfrak{L}_a : \mathfrak{L}_a \models Pn\}$. Since all subsets of $\{1,2,\dots,\mathsf{exp}_2(13n)\}$ appear as predicates of orders in $\mathcal{L}_{13n}$, quantification over subsets of $\{1,2,\dots,\mathsf{exp}_2(13n)\}$ can be reduced to quantification over elements satisfying φ^n_{lin}. We only need to construct a formula $\varphi^n_=(b,a,b',a')$ which expresses that b is the j-th element of D^n_a iff b' is the j-th element of $D^n_{a'}$. Note that this is the case if and only if the minimal path from b to a consists of the same transitions (in the same order) as the path from b' to a'. Thus, we may set

$$\begin{aligned}\varphi^n_=(x_1,x_2,y_1,y_2) :=\ & \varphi^n_{lin}(x_2) \wedge \varphi^n_{lin}(y_2) \wedge \varphi_e^{=2^{13n}}(x_1,x_2,y_1,y_2),\\ \varphi^n_\in(x,z,y) :=\ & \exists z'(\varphi^n_=(x,y,z',z) \wedge \varphi^n_P(z',z)) \text{ and}\\ I_n :=\ & (\delta^n(x,y), \varphi^n_<(x_1,x_2,y), \varphi^n_P(x,y), \varphi^n_\in(x,z,y)).\end{aligned}$$

Theorem 5.9. *There is a prescribed set M such that there is an* MSO*-to-*FO*-interpretation $(I_n)_{n\in\mathbb{N}} \subseteq M$ of $(\mathcal{L}_{13n})_{n\in\mathbb{N}}$ in* $\mathsf{NPT}(\mathcal{S})$.

Proof. Note that all formulas occuring in I_n are in prenex normal form. Moreover their size is linear in n. By moving iterative definitions to the front, we obtain an interpretation as desired. □

Corollary 5.10. *The* FO *theory of* $\mathsf{NPT}(\mathcal{S})$ *is* $\mathsf{ATIME}(\mathsf{exp}_2(cn), cn)$*-hard. Thus,* FO *model checking on the class of all* NPT *is* $\mathsf{ATIME}(\mathsf{exp}_2(cn), cn)$*-complete. Moreover, the set of* FO *sentences valid in every nested pushdown tree is* $\mathsf{ATIME}(\mathsf{exp}_2(cn), cn)$*-hard.*

6. Conclusions

We have studied the complexity of first-order model checking on the class of nested pushdown trees. We obtained a matching lower bound resulting in the fact that the first-order model checking is $\mathsf{ATIME}(\mathsf{exp}_2(cn), cn)$-complete. This bound even holds for a fixed nested pushdown tree. Thus, also the expression complexity of first order model checking is exactly in $\mathsf{ATIME}(\mathsf{exp}_2(cn), cn)$. The exact structure complexity of first-order model checking remains open. We have given an $\mathsf{ATIME}(\mathsf{exp}(cn), cn)$-algorithm[3] but we do not have any lower bounds.

Another open question concerns decidability of model checking for other fragments of monadic second order logic on nested pushdown trees. For instance, is first-order logic extended by the transitive closure operator decidable? Is the extension of first-order logic by regular reachability decidable? We know [4] that the extension of first-order logic by the reachability predicate is decidable and has non-elementary complexity. We have started to investigate the extension by transitive closure operators and we believe that it is undecidable if we allow sufficiently many nestings of the transitive closure operator.

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