(Tree-) Automatic Well-Founded Order Trees have Small Ordinal Ranks

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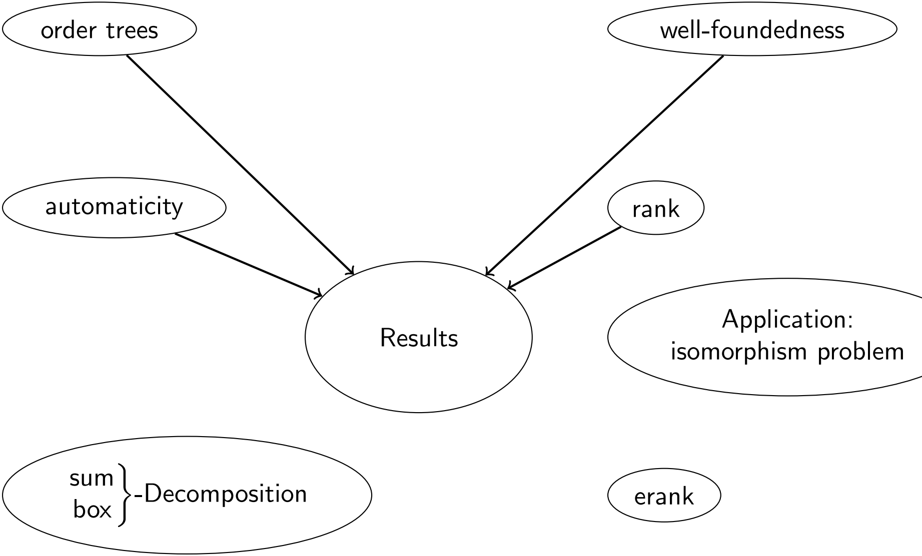
Joint work with Jiamou Liu and Markus Lohrey

Overview



order trees
well-foundedness
automaticity
rank
Results
Application:
isomorphism problem
sum
box
-Decomposition
erank

Order Trees

Definition (Partial Order)

$(P, \leq)$

- reflexive ($\forall p \quad p \leq p$),
- transitive ($\forall p, q, r \in P \quad p \leq q \leq r \Rightarrow p \leq r$),
- antisymmetric ($\forall p, q \in P \quad p \leq q \leq p \Rightarrow p = q$).

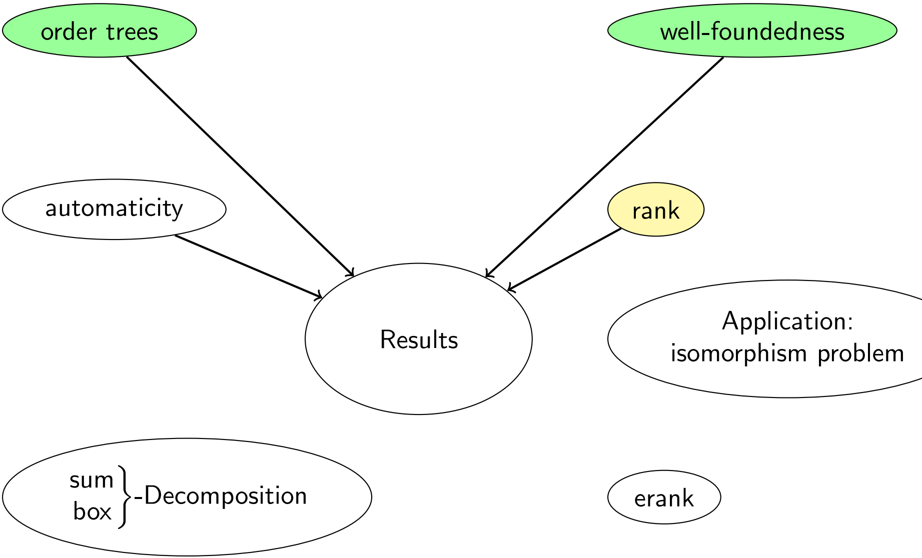
Definition (Well-foundedness)

$(P, \leq)$ partial order is *well-founded* (*wf*) if
no infinit descending chain $(p_1 > p_2 > p_3 > \dots)$

Definition (Order Forest / Tree)

$(F, \leq)$ partial order is (order) *forest* if
$\forall f \;\; F_f := \{g \in F \mid f \leq g\}$ finite and $g, h \in F_f \Rightarrow g \leq h$ or $h \leq g$.
Tree: forest with global maximum

Overview



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Rank of a Well-Founded Partial Order

Definition (Rank)

$(P, \leq)$ well-founded partial order

$$\mathrm{rank}(p, P) := \sup\{\alpha + 1 \mid \exists p' < p \ \mathrm{rank}(p') \geq \alpha\}$$
$$\mathrm{rank}(P) := \sup\{\alpha \mid \exists p \in P \ \mathrm{rank}(p, P) \geq \alpha\}$$

Intuitively:

- Each element has a higher rank than all smaller elements
- The rank is minimal with this property
- Rank of a structure = supremum of occuring ranks

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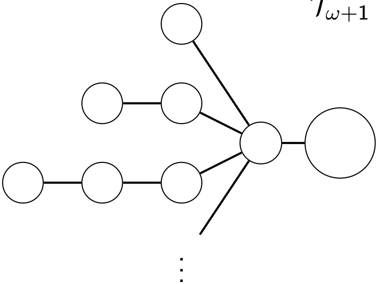
Example

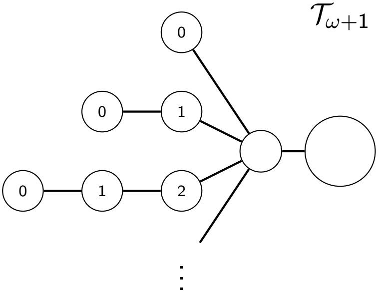
Limit ordinal λ: $\mathrm{rank}(\lambda) = \lambda$
Successor ordinal $\alpha + 1$: $\mathrm{rank}(\alpha + 1) = \alpha$

Ranks: Examples

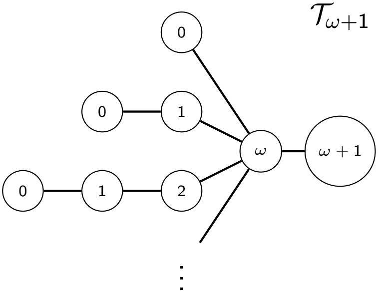


$\mathcal{T}_{\omega+1}$

Ranks: Examples



Ranks: Examples



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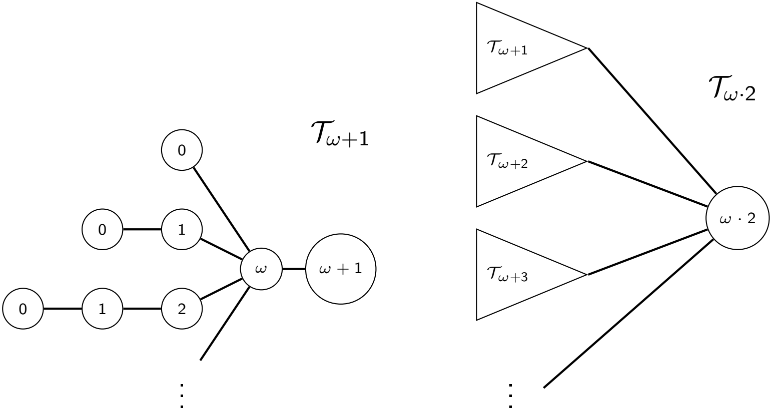

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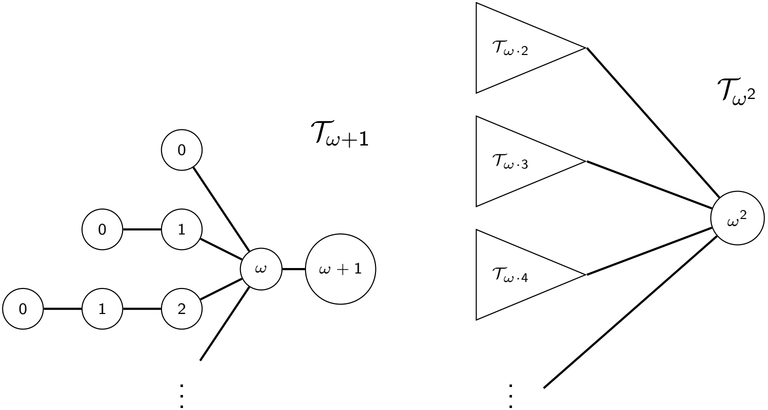

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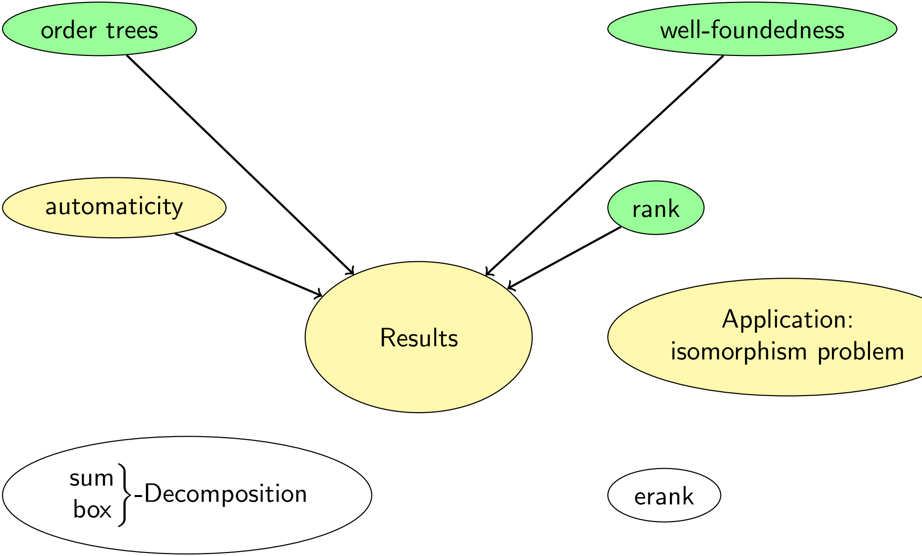
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Automatic Structures

Definition

$(P, \leq)$ *word-automatic*:
P regular language
$\{p_1 \otimes p_2 \mid p_1 \leq p_2\}$ regular language

Example

Convolution: $abc \otimes defg = \frac{a}{d}\frac{b}{e}\frac{c}{f}\frac{\square}{g}$

Definition

$(P, \leq)$ *tree-automatic*:
P regular tree language
$\{p_1 \otimes p_2 \mid p_1 \leq p_2\}$ regular tree language

Ranks of Word-Automatic Structures

Theorem (Khoussainov, Minnes 2009)

Each word-automatic well-founded partial order has rank $< \omega^\omega$.

Bound is optimal:

- ordinal $\alpha + 1 < \omega^\omega$ is word-automatic (of rank α)
- partial orders *without infinite chains* reach all ranks $< \omega^\omega$.

Theorem

Each word-automatic well-founded forest has rank $< \omega^2$.

Bound is optimal

Ranks of Tree-Automatic Structures

Theorem (Cachat 2006)

There is a tree-automatic well-founded partial order of rank $\alpha < \omega^{\omega^\omega}$.

Proof.

Ordinal $\alpha + 1 < \omega^{\omega^\omega}$ is tree-automatic of rank α. □

Theorem

Each tree-automatic well-founded forest has rank $< \omega^\omega$.

Bound is optimal

Application

Isomorphism Problem (IP)

Input: $\mathfrak{A}, \mathfrak{B}$ tree-automatic well-founded trees (Represented by Automata)
Output: $\mathfrak{A} \simeq \mathfrak{B}$?

Theorem

IP for tree-automatic wf trees: complete for $\Delta^0_{\omega^\omega} = \Sigma^0_{\omega^\omega} \cap \Pi^0_{\omega^\omega}$

Proof.

Construct formulas iso_α:
$\mathfrak{A} \cup \mathfrak{B} \models \mathrm{iso}_\alpha(r_a, r_b)$ iff $\mathrm{rank}(\mathfrak{A}), \mathrm{rank}(\mathfrak{B}) \leq \alpha$ and $\mathfrak{A} \simeq \mathfrak{B}$.

□

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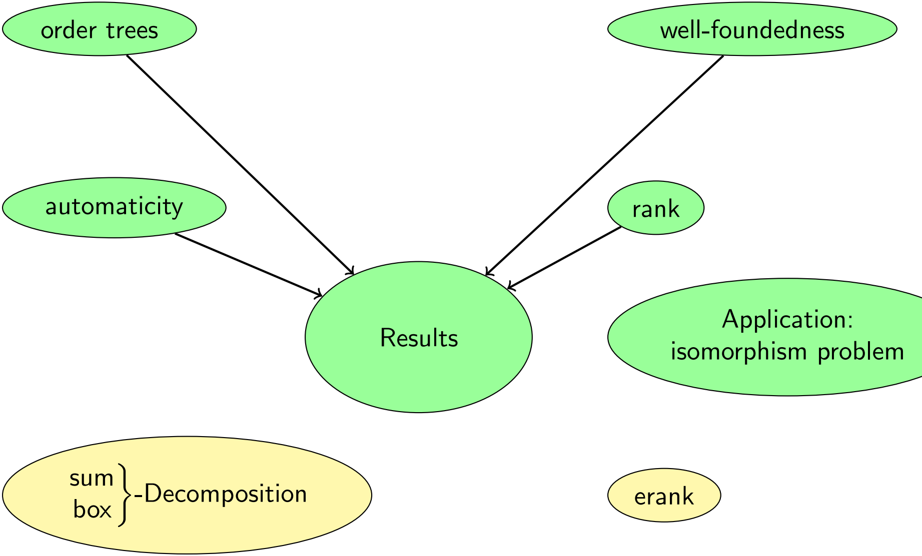
r_a / r_b root of $\mathfrak{A}/\mathfrak{B}$; E_x: children of x

$$\mathrm{iso}_\alpha = \forall x \in E_{r_a} \cup E_{r_b} \Big(\bigvee_{\beta<\alpha} \mathrm{iso}_\beta(x,x) \wedge \\ (\exists^{\geq k} y \in E_{r_a} \mathrm{iso}_\beta(x,y) \Leftrightarrow \exists^{\geq k} y \in E_{r_b} \mathrm{iso}_\beta(x,y))\Big)$$

$\Sigma^0_{\omega^\omega}$-formula: $\mathrm{iso}_{\omega^\omega} = \bigvee_{\alpha < \omega^\omega} \mathrm{iso}_\alpha$ ($\Pi^0_{\omega^\omega}$ similar) □

Overview



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Erank

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Definition (Erank of well-founded partial order)

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$$\mathrm{erank}(p, P) := \sup\{\alpha + 1 \mid \exists^{\infty} p' < p \ \mathrm{erank}(p') \geq \alpha\}$$
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- Erank of Forest $(F, \leq)$ = rank of infinitely branching subforest
- *embedding rank*: $\mathrm{erank}(F) \geq i \Leftrightarrow \mathbb{N}^{\leq i} \hookrightarrow (F, \leq)$

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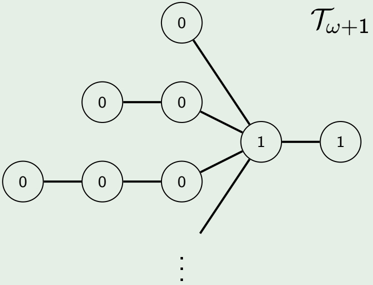
Example

Ordinal $\omega^{\alpha+1} + 1$ has erank ω^{α}

Examples

Example



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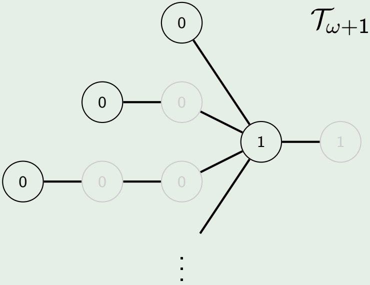

$\mathcal{T}_{\omega+1}$

Examples

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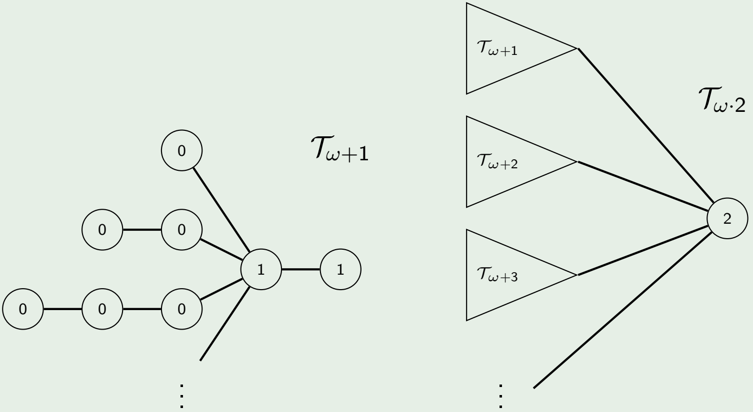

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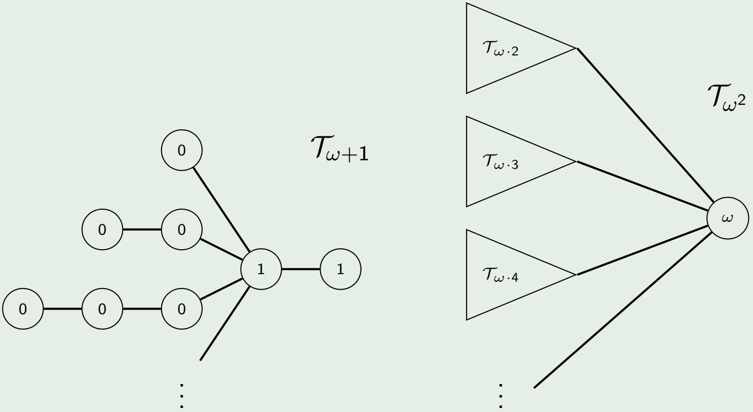

Proof of the Word-Automata Result

Lemma

erank$(P) \leq$ rank$(P) < \omega \cdot$ erank$(P) + \omega$

Proof.

Straightforward transfinite induction □

Lemma (Kuske, Liu,Lohrey; to appear)

$(F, \leq)$ *word-automatic well-founded forest* $\Rightarrow$ erank$(F) < \omega$

Theorem

Each word-automatic well-founded forest has rank $< \omega^2$*.*

Decomposition Technique for Tree-Automatic Structures

$\mathfrak{F} = (F, \leq)$ tree-automatic structure

$\varphi(x, y)$ formula with parameter y

$\mathfrak{F}_p := \mathfrak{F}\restriction_{\{f \in F \mid \mathfrak{F} \models \varphi(f,p)\}}$

Theorem (Delhomme 2004, Kartzow/Huschenbett 2011)

$\exists \mathfrak{B}_1, \dots, \mathfrak{B}_n$ *tree-automatic s.t.* $\forall p \in F$
$\mathfrak{F}_p$ *is a sum-augmentation of nice box-augmentations of* $\mathfrak{B}_1, \dots, \mathfrak{B}_n$

Sum-Augmentations

Theorem (Delhomme 2004, Kartzow/Huschenbett 2011)

$\exists\mathfrak{B}_1, \ldots, \mathfrak{B}_n$ tree-automatic s.t. $\forall p \in F$
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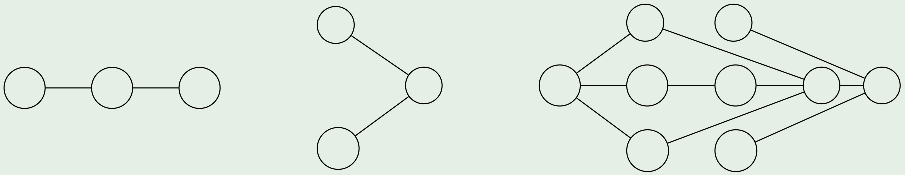
Definition

$\mathfrak{F}$ is sum-agumentation of $\mathfrak{B}_1, \ldots \mathfrak{B}_n$ if
$\mathfrak{F} = \mathfrak{B}_{i_1} \sqcup \cdots \sqcup \mathfrak{B}_{i_m}$ + additional edges between the substructures

Example



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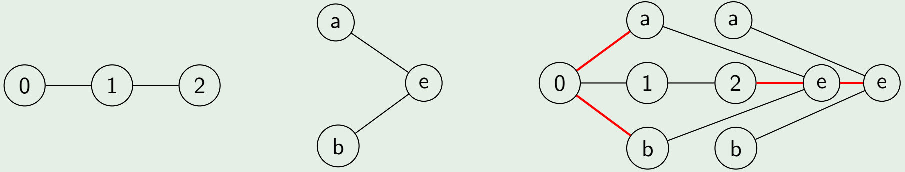

Box-Augmentations

Theorem (Delhomme 2004, Kartzow/Huschenbett 2011)

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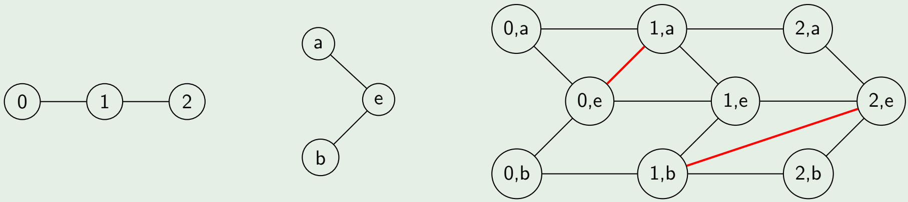
$\mathfrak{F} = \mathfrak{B}_{i_1} \times \cdots \times \mathfrak{B}_{i_m}$ + new diagonal edges

Example



Lemma

$\mathfrak{F}$ *well-founded forest;* $\text{erank}(\mathfrak{F}) = \omega^\alpha$
$\mathfrak{F}$ *sum-augmentation / nice box-augmentation of* $\mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_n$
$\Rightarrow \exists i \quad \text{erank}(\mathfrak{B}_i) = \omega^\alpha$

Proof.

Sums: $\text{erank}(\mathfrak{F}) \leq \text{erank}(\mathfrak{B}_{i_1}) \oplus \text{erank}(\mathfrak{B}_{i_2}) \oplus \cdots \oplus \text{erank}(\mathfrak{B}_{i_m})$

□

Natural Sum $\oplus$

Addition on Cantor normal forms as polynomials in ω

$$(\omega^{\alpha_1} \cdot c_1 + \cdots + \omega^{\alpha_n} \cdot c_n) \oplus (\omega^{\alpha_1} \cdot d_1 + \cdots + \omega^{\alpha_n} \cdot d_n) = \omega^{\alpha_1} \cdot (c_1 + d_1) + \cdots + \omega^{\alpha_n} \cdot (c_n + d_n)$$

Corollary

$\omega^\alpha \leq \beta_1 \oplus \beta_2$ *implies* $\beta_1 \geq \omega^\alpha$ *or* $\beta_2 \geq \omega^\alpha$.

Lemma

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Boxes: nice Box-augmentation = sum-augmentation of similar forests □

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$\mathfrak{F}$ *sum-augmentation / nice box-augmentation of* $\mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_n$
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$\mathfrak{F}_p$ *is a sum-augmentation of nice box-augmentations of* $\mathfrak{B}_1, \dots, \mathfrak{B}_n$

$\varphi := x \leq y$: $\{\mathfrak{F}_p : p \in \mathfrak{F}\}$ contains all subtrees
Assume $\mathfrak{T}$ tree-automatic of erank $\geq \omega^\omega$
$\Rightarrow \forall i \; \exists \; \mathfrak{F}_p$ with $\mathrm{erank}(\mathfrak{F}_p) = \omega^i$
$\Rightarrow \forall i \; \exists j \quad \mathrm{erank}(\mathfrak{B}_j) = \omega^i$; $i \in \mathbb{N}$, but $1 \leq j \leq n$

Proof of the Tree-Automata Result

Lemma

Each tree-automatic well-founded forest has erank $< \omega^\omega$.

Lemma

$\text{erank}(P) \leq \text{rank}(P) < \omega \cdot \text{erank}(P) + \omega$

Theorem

Each tree-automatic well-founded forest has rank $< \omega^\omega$.

Proof.

$erank(\mathfrak{F}) = \alpha < \omega^\omega$

$\Rightarrow \exists i \ \alpha \leq \omega^i$

$\Rightarrow \text{rank}(\mathfrak{F}) < \omega \cdot \alpha + \omega \leq \omega \cdot \omega^i + \omega \leq \omega^{i+2} < \omega^\omega$ □

Summary

- Ranks of well-founded

	tree-aut.	word-aut.
partial orders	$< \omega^{\omega^\omega}$	$< \omega^\omega$
forests	$< \omega^\omega$	$< \omega^2$

- Isomorphism Problem for tree-automatic well-founded trees: $\Delta^0_{\omega^\omega}$ complete (under Turing-reductions)
- Proof: improved Sum/ Box decomposition technique a la Delhommé

Open Problems:

- compute erank/rank of tree-automatic well-founded tree
- maximal erank of tree-automatic partial order *without infinite chains*