Ranks of Tree-Automatic Well-Founded Order Trees

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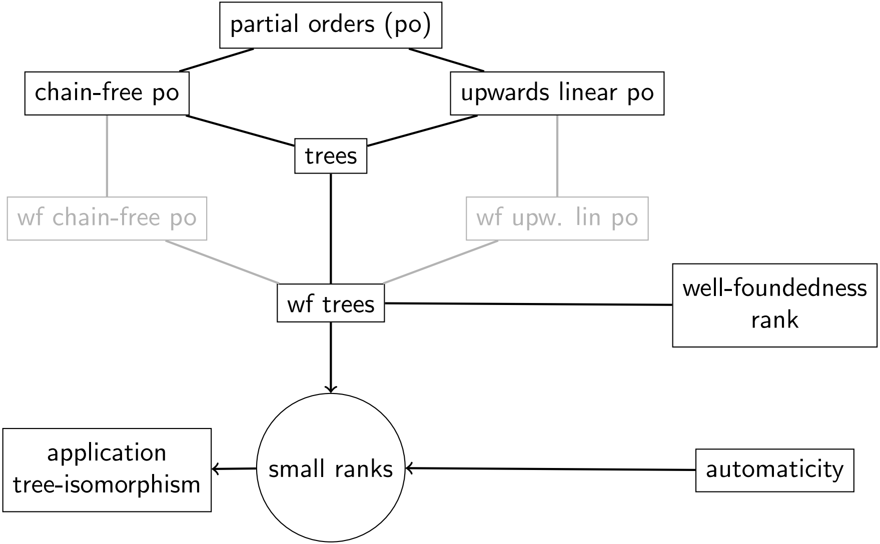
joint work with Jiamou Liu and Markus Lohrey

Overview


partial orders (po)
chain-free po
upwards linear po
trees
wf chain-free po
wf upw. lin po
wf trees
well-foundedness rank
small ranks
application tree-isomorphism
automaticity

Order Trees

Definition (Partial Order)

$(P, \leq)$

- reflexive $(\forall p \quad p \leq p)$,
- transitive $(\forall p, q, r \in P \quad p \leq q \leq r \Rightarrow p \leq r)$,
- antisymmetric $(\forall p, q \in P \quad p \leq q \leq p \Rightarrow p = q)$.

Definition

$(P, \leq)$ is upwards linear if $\forall p \in P \quad \{p' \mid p \leq p'\}$ is linear
$(P, \leq)$ is chain-free if there is no infinite ascending chain $(p_1 < p_2 < \dots)$.

Definition (Order Forest / Tree)

$(F, \leq)$ is (order) forest if $(F, \leq)$ is upwards linear and chain-free
Tree: forest with global maximum

Rank of a Well-Founded Partial Order

Definition (Well-foundedness)

$(P, \leq)$ partial order is *well-founded* (*wf*) if
no infinit descending chain $(p_1 > p_2 > p_3 > \dots)$

Definition (Rank)

$(P, \leq)$ well-founded partial order

$$\mathrm{rank}(p, P) := \sup\{\mathrm{rank}(p') + 1 \mid p' < p\}$$
$$\mathrm{rank}(P) := \sup\{\mathrm{rank}(p, P) \mid p \in P\}$$

Intuitively:

- Each element has a higher rank than all smaller elements
- The rank is minimal with this property
- Rank of partial order = supremum of occuring ranks

Rank of a Well-Founded Partial Order

Definition (Well-foundedness)

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Definition (Rank)

$(P, \leq)$ well-founded partial order

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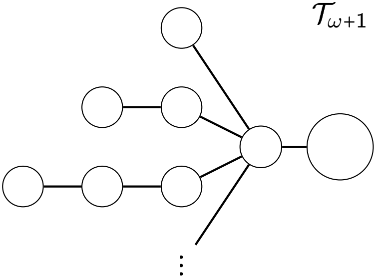
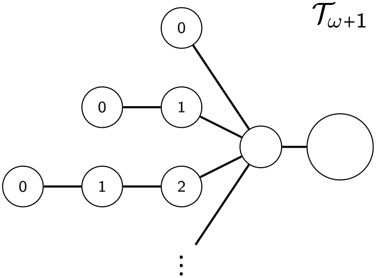
Example

Limit ordinal λ: $\mathrm{rank}(\lambda) = \lambda$
Successor ordinal $\alpha + 1$: $\mathrm{rank}(\alpha + 1) = \alpha$

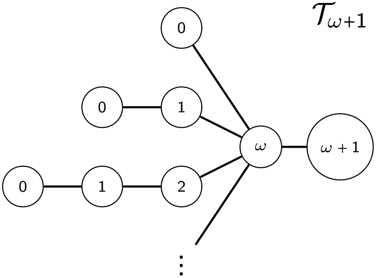
Ranks: Examples



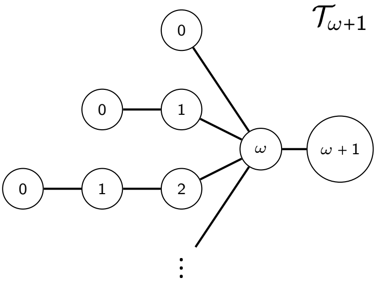
Ranks: Examples

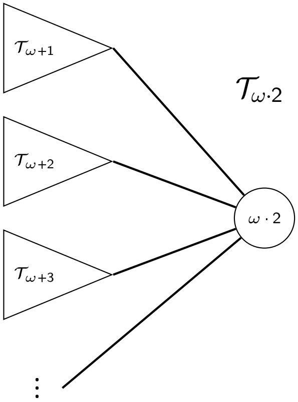


Ranks: Examples



Ranks: Examples





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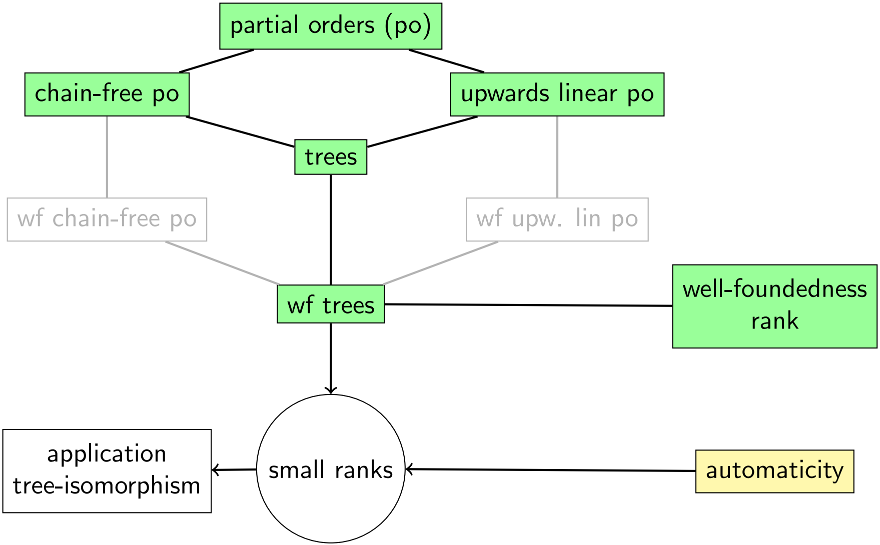
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Automatic Structures

Definition

$(P, \leq)$ *word-automatic*:

- P regular language (= accepted by DFA)
- $\{(p_1, p_2) \in P^2 \mid p_1 \leq p_2\}$ accepted by synchonous two-tape DFA

Definition

$(P, \leq)$ *tree-automatic*:
P regular tree language
$\{(p_1, p_2) \in P^2 \mid p_1 \leq p_2\}$ regular tree language

Theorem (decidability of FO model checking)

Given an automatic structure $(P, \leq)$ *and an FO-sentence* φ
$(P, \leq) \vDash \varphi$*? decidable*

Ranks of Word-Automatic Structures

Theorem (Delhommé 2004)

Each word-automatic well-founded partial order has rank $< \omega^\omega$.

Bound is optimal:

- ordinal $\alpha + 1 < \omega^\omega$ is word-automatic (of rank α)

Theorem

Each word-automatic well-founded forest has rank $< \omega^2$.

Bound is optimal

Ranks of Tree-Automatic Structures

Theorem (Delhommé 2004)

The ordinal α is tree-automatic iff $\alpha < \omega^{\omega^\omega}$.

Conjecture

Every tree-automatic well-founded partial order has rank below $\alpha < \omega^{\omega^\omega}$.

Theorem

Each tree-automatic well-founded forest has rank $< \omega^\omega$.

Bound is optimal

Application

Isomorphism Problem (IP)

Input: $\mathfrak{A}, \mathfrak{B}$ tree-automatic well-founded trees (represented by automata)
Output: $\mathfrak{A} \simeq \mathfrak{B}$?

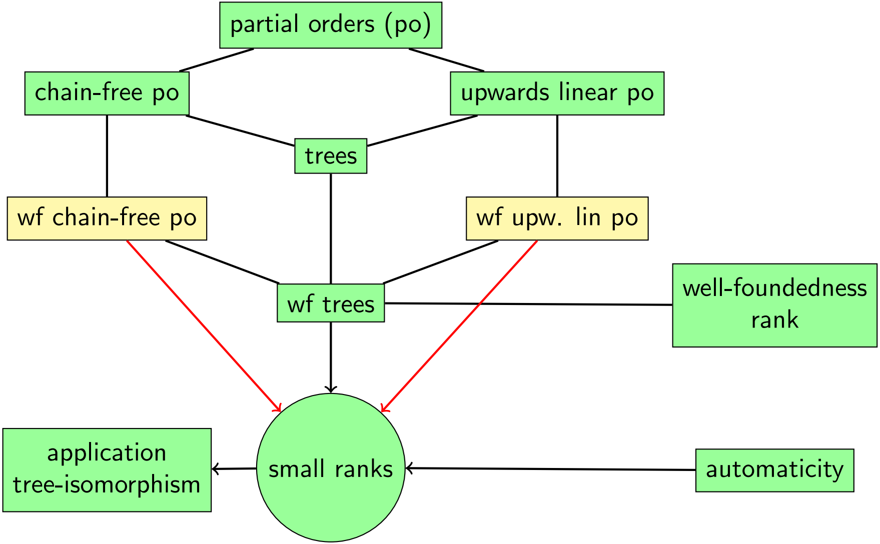
Theorem

IP for tree-automatic wf trees is complete for $\Delta^0_{\omega^\omega} = \Sigma^0_{\omega^\omega} \cap \Pi^0_{\omega^\omega}$.

Proof.

Isomorphism of trees of rank at most $\alpha \to \Sigma^0_\alpha$-formula □

Overview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What makes forests special?

Recall

A partial order is a forest iff

1. it is upwards linear and
2. chain-free.

Problem

Ranks of upwards linear / chain-free tree-automatic partial orders?

Upwards Linear Wf Partial Orders

Lemma

Tree-automatic upwards linear wf partial orders realise all ranks $< \omega^{\omega^\omega}$.

Proof.

Ordinals are upwards linear! □

Lemma

Every tree-automatic upwards linear wf partial orders has rank $< \omega^{\omega^\omega}$.

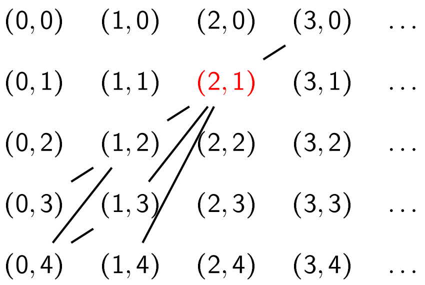
Chain-Free Partial Orders

Definition

$\mathcal{N} := (\mathbb{N} \times \mathbb{N}, \prec)$ with $(a_1, b_1) \prec (a_2, b_2)$ iff $a_1 < a_2$ and $b_1 > b_2$

Lemma

$\mathcal{N}$ is word-automatic, $\text{rank}(\mathcal{N}) = \omega$ *and chain-free.*



Chain-Free Partial Orders 2

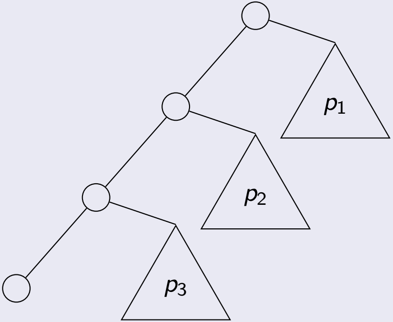
Lemma

$\mathcal{P}$ chain-free tree-aut. $\Rightarrow$
$\bigcup_{i\in\mathbb{N}} \mathcal{P}^i$ *chain-free tree-aut.; Rank:* $\mathrm{rank}(\mathcal{P})^\omega$.

Order on P^i: lexicographic order on P^i

Proof.

Encoding of (p_1, p_2, p_3) Encoding of (q_1, q_2, q_3)



□

Chain-Free Partial Orders 2

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Order on P^i: lexicographic order on P^i

Corollary

$\mathcal{N}^0 := \mathcal{N}$
$\mathcal{N}^{j+1} := \bigcup_{i\in\mathbb{N}} (\mathcal{N}^j)^i$

1. $\mathcal{N}^j$ *has rank* ω^{ω^j},
2. *is tree-automatic and*
3. *chain-free.*

Summary

tree-automatic well-founded	realised ranks	upper bound
partial orders	$\alpha < \omega^{\omega^\omega}$	?
chain-free partial orders	$\alpha < \omega^{\omega^\omega}$	?
upwards linear partial orders	$\alpha < \omega^{\omega^\omega}$	ω^{ω^ω}
forests	$\alpha < \omega^\omega$	ω^ω

- Isomorphism Problem for tree-automatic well-founded trees: $\Delta^0_{\omega^\omega}$ complete (under Turing-reductions)

Open Problems:

- Conjecture: ? = ω^{ω^ω}
- Characterise all tree-automatic wf trees / wf partial orders
- compute rank of tree-automatic well-founded tree